

# REGULAR BASES AT NON-ISOLATED POINTS AND METRIZATION THEOREMS

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**ABSTRACT.** In this paper, we define the spaces with a regular base at non-isolated points and discuss some metrization theorems. We firstly show that a space  $X$  is a metrizable space, if and only if  $X$  is a regular space with a  $\sigma$ -locally finite base at non-isolated points, if and only if  $X$  is a perfect space with a regular base at non-isolated points, if and only if  $X$  is a  $\beta$ -space with a regular base at non-isolated points. In addition, we also discuss the relations between the spaces with a regular base at non-isolated points and some generalized metrizable spaces. Finally, we give an affirmative answer for a question posed by F. C. Lin and S. Lin in [7], which also shows that a space with a regular base at non-isolated points has a point-countable base.

## 1. INTRODUCTION

The bases of topological spaces occupy a core position in the study of the topological theories and metrization problems, which has produced many kinds of metrization theorems, and establishes a foundation for the topological development [12]. For example, the following is a classic metrization theorem.

**Theorem 1.1.** *The following are equivalent for a space  $X$ :*

- (1)  $X$  is metrizable;
- (2)  $X$  is a  $T_1$ -space with a regular base;
- (3)  $X$  is a regular space with a  $\sigma$ -locally finite base.

In recent years, the theory of regular bases in topological spaces played an important role in generalized metrizable spaces [2, 17]. On the other hand, in the study of the theories of topological spaces, we are mainly concerned with the properties of neighborhoods on non-isolated points, and also discuss the relation between their properties and global properties. For example, a study of spaces with a sharp base, a weakly uniform base or an uniform base at non-isolated points [2, 3, 7] shows that some properties of a non-isolated point set of a topological space will help us discuss the global construction of a space. Especially, a space  $X$  with a uniform base at non-isolated points if and only if  $X$  is the open and boundary-compact image of a metric space [7]. The most typical example is the spaces obtained from a metrizable space by isolating the points of a subset.

Let  $\mathcal{B}$  be a base for a space  $X$ . For any  $x \in X$ , the base  $\mathcal{B}$  of  $X$  is called *regular* at a point  $x$  if, for every neighborhood  $U$  of  $x$ , there exists an open subset  $V$  such that  $x \in V \subset U$  and  $\{B \in \mathcal{B} : B \cap V \neq \emptyset \text{ and } B \not\subset U\}$  is finite.

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2000 *Mathematics Subject Classification.* 54D70; 54E35; 54D20.

*Key words and phrases.* Metrization; regular bases; locally-finite families;  $\beta$ -spaces; proto-metrizable; discretization; point-regular bases.

Supported by the NSFC(No. 10971185) and the Educational Department of Fujian Province(No. JA09166).

By Theorem 1.1, every metric space has a base which is regular at non-isolated points. However, there exists a non-metrizable space with a base which is regular at non-isolated points, see the following Example 1.2.

*Example 1.2.* Let  $X$  be the closed unit interval  $\mathbb{I} = [0, 1]$  and  $B$  a Bernstein subset of  $I$ . In other words,  $B$  is an uncountable set which contains no uncountable closed subset of  $I$ . Endow  $X$  with the following topology, i.e., Michael line [15]:  $G$  is an open subset for  $X$  if and only if  $G = U \cup Z$ , where  $U$  is an open subset of  $\mathbb{I}$  with Euclidean topology and  $Z \subset B$ . Let  $\mathcal{B}$  be a base of  $\mathbb{I}$  with the Euclidean topology, where  $\mathcal{B}$  is regular at every point of  $\mathbb{I}$ . Then  $\mathcal{P} = \mathcal{B} \cup \{\{x\} : x \in B\}$  is a base for  $X$  and also regular at non-isolated points.

Hence this causes our interests in a study of spaces with a base which is regular at non-isolated points, and the related problems of the metrizability. In this paper, we shall prove that spaces with a regular base at non-isolated points are strictly between the discretizations of metrizable spaces and proto-metrizable spaces, and we also obtain some metrization theorems which help us to better understand the relation between the properties at non-isolated points and global properties in the study the generalized metrizable spaces.

In this paper all spaces are  $T_1$  unless it is explicitly stated which separation axiom is assumed, and all maps are continuous and onto. By  $\mathbb{R}, \mathbb{N}$ , denote the set of real numbers and positive integers, respectively. For a space  $X$ , let  $I = I(X) = \{x : x \text{ is an isolated point of } X\}$  and  $\mathcal{I}(X) = \{\{x\} : x \in I(X)\}$ . Let  $\mathcal{P}$  be a family of subsets for  $X$ , and we denote

$$\begin{aligned} \text{st}(x, \mathcal{P}) &= \cup\{P \in \mathcal{P} : x \in P\}, x \in X; \\ \text{st}(A, \mathcal{P}) &= \cup\{P \in \mathcal{P} : A \cap P \neq \emptyset\}, A \subset X; \\ \mathcal{P}^m &= \{P \in \mathcal{P} : \text{if } P \subset Q \in \mathcal{P}, \text{ then } Q = P\}. \end{aligned}$$

Readers may refer to [6, 13] for unstated definitions and terminology.

## 2. REGULAR BASES AT NON-ISOLATED POINTS

**Definition 2.1.** Let  $\mathcal{B}$  be a base of a space  $X$ .  $\mathcal{B}$  is a *regular base*, see e.g. [6] (*regular base at non-isolated points*, resp.) for  $X$  if for each (non-isolated, resp.) point  $x \in X$ ,  $\mathcal{B}$  is regular at  $x$ .

It is obvious that regular bases  $\Rightarrow$  regular bases at non-isolated points, but regular bases at non-isolated points  $\nRightarrow$  regular bases by Example 1.2.

**Definition 2.2.** Let  $\{\mathcal{W}_i\}_{i \in \mathbb{N}}$  be a sequence of open covers of a space  $X$  and  $\mathcal{I}(X) \subset \bigcup_{i \in \mathbb{N}} \mathcal{W}_i$ .  $\{\mathcal{W}_i\}_{i \in \mathbb{N}}$  is called a *strong development*, see e.g. [6] (*strong development at non-isolated points*, resp.) for  $X$  if for every  $x \in X$  ( $x \in X - I$ ) and each neighborhood  $U$  of  $x$  there exist a neighborhood  $V$  of  $x$  and an  $i \in \mathbb{N}$  such that  $\text{st}(V, \mathcal{W}_i) \subset U$ . If  $\{\mathcal{W}_i\}_{i \in \mathbb{N}}$  is a strong development at non-isolated points, then so is  $\{\mathcal{W}_i \cup \mathcal{I}(X)\}_{i \in \mathbb{N}}$ .

The following Lemma 2.3 is proved similarly to Lemma 5.4.3 in [6], and leave to the reader the easy proofs of Lemma 2.4 and 2.5.

**Lemma 2.3.** *If  $\mathcal{B}$  is a regular base at non-isolated points for a space  $X$ , then the family  $\mathcal{B}^m \subset \mathcal{B}$  is locally finite at non-isolated points and also covers  $X - I$ .*

**Lemma 2.4.** *Let  $\mathcal{B}$  be a regular base at non-isolated points for  $X$ . If  $\mathcal{B}' \subset \mathcal{B}$  is point-finite at non-isolated points, then  $\mathcal{B}'' = (\mathcal{B} - \mathcal{B}') \cup \mathcal{I}(X)$  is a regular base at non-isolated points for  $X$ .*

**Lemma 2.5.** *If  $\mathcal{B}$  is a regular base at non-isolated points for  $X$ , put*

$$\mathcal{B}_1 = \mathcal{B}^m, \quad \mathcal{B}_i = [(\mathcal{B} - \bigcup_{j=1}^{i-1} \mathcal{B}_j) \cup \mathcal{I}(X)]^m, i = 2, 3, \dots.$$

*Then  $\mathcal{B} = (\bigcup_{i=1}^{\infty} \mathcal{B}_i) \cup \mathcal{I}(X)$ , and for each  $i \in \mathbb{N}$ ,  $\mathcal{B}_i$  is locally finite at non-isolated points and  $\mathcal{B}_{i+1} \cup \mathcal{I}(X)$  refines  $\mathcal{B}_i \cup \mathcal{I}(X)$ .*

Recall that a topological space  $X$  is *monotonically normal* [10] if for each ordered pair  $(p, C)$ , where  $C$  is a closed set for  $X$  and  $p \in X - C$ , there exists an open subset  $H(p, C)$  satisfying the following conditions:

- (i)  $p \in H(p, C) \subset X - C$ ;
- (ii) For every closed subset  $D$  for  $X$ , if  $D \subset C$ , then  $H(p, C) \subset H(p, D)$ ;
- (iii) If  $p \neq q \in X$ , then  $H(p, \{q\}) \cap H(q, \{p\}) = \emptyset$ .

A  $T_2$ -paracompact space or monotonically normal space is a collectionwise normal space [10].

**Lemma 2.6.** *If a space  $X$  has a strong development at non-isolated points, then  $X$  is a monotonically normal and paracompact space.*

*Proof.* Let  $\{\mathcal{W}_i\}_{i \in \mathbb{N}}$  be a strong development at non-isolated points for  $X$ , where  $\mathcal{W}_{i+1}$  refines  $\mathcal{W}_i$  for every  $i \in \mathbb{N}$ .

(1) Claim. Let  $A$  be a closed subset for  $X$ . If  $x \in (X - A) \cap (X - I)$ , then there exists an  $i \in \mathbb{N}$  such that  $\text{st}(x, \mathcal{W}_i) \cap \text{st}(A, \mathcal{W}_i) = \emptyset$ .

In fact, since  $X - A$  is an open neighborhood of  $x$ , there exists a  $j \in \mathbb{N}$  and an open neighborhood  $V$  of  $x$  such that  $\text{st}(V, \mathcal{W}_j) \subset X - A$ . Also, there exists a  $i \geq j$  such that  $\text{st}(x, \mathcal{W}_i) \subset V$ . Since  $\text{st}(A, \mathcal{W}_i) \subset X - V$ , we have  $\text{st}(x, \mathcal{W}_i) \cap \text{st}(A, \mathcal{W}_i) = \emptyset$ .

(2)  $X$  is a monotonically normal space.

Let  $C$  be a closed subset for  $X$  and  $p \in X - C$ . If  $p \in I$ , then we let  $H(p, C) = \{p\}$ ; if  $p \in X - I$ , then there exists a minimum  $n \in \mathbb{N}$  such that  $\text{st}(p, \mathcal{W}_n) \cap \text{st}(C, \mathcal{W}_n) = \emptyset$  by (1), so we let  $H(p, C) = \text{st}(p, \mathcal{W}_n)$ . Then  $H(p, C)$  is an open subset for  $X$ . Clearly this definition of  $H(p, C)$  satisfies the conditions (i) and (ii) in the above definition of monotonically normal spaces. We next prove that it also satisfies (iii). In fact, for any distinct points  $p, q$  in  $X - I$ , fix the  $n, m$  for which:

$$H(p, \{q\}) = \text{st}(p, \mathcal{W}_n) \text{ and } H(q, \{p\}) = \text{st}(q, \mathcal{W}_m).$$

Then

$$\text{st}(p, \mathcal{W}_n) \cap \text{st}(q, \mathcal{W}_n) = \emptyset \text{ and } \text{st}(p, \mathcal{W}_m) \cap \text{st}(q, \mathcal{W}_m) = \emptyset.$$

By the choice of  $n, m$ , we have  $n = m$ , i.e.  $H(p, \{q\}) \cap H(q, \{p\}) = \emptyset$ . Hence it also satisfies (iii) in the definition of monotonically normal spaces.

(3)  $X$  is a paracompact space.

Let  $\{G_s\}_{s \in S}$  be an open cover for  $X$  and  $S_0 = \{s \in S : G_s \cap (X - I) \neq \emptyset\}$ . Fix a well-order by “ $<$ ” on  $S_0$ . For every  $i \in \mathbb{N}$ ,  $s \in S_0$ , put

$$F_{s,i} = X - (\text{st}(X - G_s, \mathcal{W}_i) \cup (\bigcup_{s' < s} G_{s'})),$$

then  $F_{s,i} \subset G_s$ .

(3.1) The closed family  $\{F_{s,i}\}_{s \in S_0, i \in \mathbb{N}}$  covers  $X - I$ .

Indeed, for every  $x \in X - I$ , there exists a minimum  $s(x) \in S_0$  such that  $x \in G_{s(x)}$ . Since  $\{\mathcal{W}_i\}_{i \in \mathbb{N}}$  is a strong development at non-isolated points for  $X$ , there exists an  $i(x) \in \mathbb{N}$  such that  $\text{st}(x, \mathcal{W}_{i(x)}) \subset G_{s(x)}$ . Hence  $x \in F_{s(x), i(x)}$ .

(3.2) For every  $i \in \mathbb{N}$ ,  $\{F_{s,i}\}_{s \in S_0}$  is a discrete and closed family for  $X$ .

The family  $\{F_{s,i}\}_{s \in S_0}$  is disjoint by construction, hence if  $x \in I$  then  $\{x\}$  is a neighborhood that intersects  $F_{s,i}$  for at most one  $s$ . If  $x \in X \setminus I$  then, using (3.1),  $x \in \bigcup_{s \in S_0} G_s$ . Hence there exists a minimum  $s(x) \in S_0$  such that  $x \in G_{s(x)}$ . Then  $G_{s(x)} \cap \text{st}(x, \mathcal{W}_i)$  is an open neighborhood of  $x$ . If  $s' < s(x)$ , then  $x \in X - G_{s'}$ , so we have

$$\text{st}(x, \mathcal{W}_i) \subset \text{st}(X - G_{s'}, \mathcal{W}_i) \text{ and } \text{st}(x, \mathcal{W}_i) \cap F_{s',i} = \emptyset;$$

If  $s' > s(x)$ , then  $G_{s(x)} \cap F_{s',i} = \emptyset$ , so there is only one member of  $\{F_{s,i}\}_{s \in S_0}$  which meets  $G_{s(x)} \cap \text{st}(x, \mathcal{W}_i)$ . Hence  $\{F_{s,i}\}_{s \in S_0}$  is a discrete and closed family for  $X$ .

$X$  is collectionwise normal since monotonically normal spaces are collectionwise normal [10]. For every  $F_{s,i}$ , there exists an open subset  $G_{s,i}$  such that  $F_{s,i} \subset G_{s,i} \subset G_s$  and  $\{G_{s,i}\}_{s \in S_0}$  is a discrete family. Let

$$\mathcal{B}_i = \{G_{s,i}\}_{s \in S_0} \cup \{\{x\} : x \in I - \bigcup_{s \in S_0} G_{s,i}\}.$$

Then  $\bigcup_{i \in \mathbb{N}} \mathcal{B}_i$  is a  $\sigma$ -locally finite open cover for  $X$  and refines  $\{G_s\}_{s \in S}$ . Since  $X$  is regular,  $X$  is paracompact.  $\square$

Next we shall prove the main theorems in this section.

**Theorem 2.7.** *A space  $X$  has a regular base at non-isolated points if and only if  $X$  has a strong development at non-isolated points.*

*Proof.* Necessity. Since  $X$  has a regular base at non-isolated points,  $X$  has a regular base at non-isolated points  $\mathcal{B} = (\bigcup_{i \in \mathbb{N}} \mathcal{B}_i) \cup \mathcal{I}(X)$  satisfying Lemma 2.5, where  $\mathcal{B}_i$  is locally finite at non-isolated points and  $\mathcal{B}_{i+1} \cup \mathcal{I}(X)$  refines  $\mathcal{B}_i \cup \mathcal{I}(X)$  for every  $i \in \mathbb{N}$ . Put  $\mathcal{W}_i = \mathcal{B}_i \cup \mathcal{I}(X)$ . We will show that  $\{\mathcal{W}_i\}_{i \in \mathbb{N}}$  is a strong development at non-isolated points for  $X$ . In fact, for every  $x \in X - I$  and each open neighborhood  $U$  of  $x$ , since  $\mathcal{B}$  is regular at non-isolated points, there exists an open neighborhood  $V \subset U$  of  $x$  such that the set of all members of  $\mathcal{B}$  that meet both  $V$  and  $X - U$  is finite. We can denote these finite elements by  $B_1, B_2, \dots, B_k$ . Then there exists a  $j \in \mathbb{N}$  such that  $\mathcal{B}_j \cap \{B_i : i \leq k\} = \emptyset$ . Hence  $\text{st}(V, \mathcal{W}_j) \subset U$ .

Sufficiency. Let  $\{\mathcal{W}_i\}_{i \in \mathbb{N}}$  be a strong development at non-isolated points for  $X$ . By Lemma 2.6,  $X$  is paracompact. For every  $i \in \mathbb{N}$ , let  $\mathcal{B}_i$  be a locally finite open refinement for  $\mathcal{W}_i$ . Without loss of generality, we may assume  $\mathcal{B}_{i+1}$  refines  $\mathcal{B}_i$  for every  $i \in \mathbb{N}$ . We next prove that  $\mathcal{B} = (\bigcup_{i \in \mathbb{N}} \mathcal{B}_i) \cup \mathcal{I}(X)$  is a regular base at non-isolated points for  $X$ . Obviously  $\mathcal{B}$  is a base for  $X$ . For every  $x \in X - I$  and each open neighborhood  $U$  of  $x$ , there exist an open neighborhood  $V$  of  $x$  and an  $i \in \mathbb{N}$  such that  $\text{st}(V, \mathcal{W}_i) \subset U$ . If  $j \geq i$ , then

$$\text{st}(V, \mathcal{B}_j) \subset \text{st}(V, \mathcal{B}_i) \subset \text{st}(V, \mathcal{W}_i) \subset U.$$

However, since each  $\mathcal{B}_j$  is locally finite, there exists an open neighborhood  $W(x)$  of  $x$  such that the set of all members of  $\bigcup_{j < i} \mathcal{B}_j$  that meet  $W(x)$  is finite. Let  $V_1 = V \cap W(x)$ . Then the set of all members of  $\mathcal{B}$  that meet  $V_1$  and  $X - U$  is finite.  $\square$

Similar to definition 2.2, we say a space  $X$  has a *development at non-isolated points* [7] if there exists a sequence  $\{\mathcal{W}_i\}_{i \in \mathbb{N}}$  of open covers for  $X$  such that, for every  $x \in X - I$  and each open neighborhood  $U$  of  $x$ , there exist an open neighborhood  $V$  of  $x$  and an  $i \in \mathbb{N}$  such that  $\text{st}(V, \mathcal{W}_i) \subset U$ .

**Theorem 2.8.** *A space  $X$  has a regular base at non-isolated points if and only if  $X$  is a  $T_2$ -paracompact space with a development at non-isolated points.*

*Proof.* Necessity. By Lemma 2.6 and Theorem 2.7, if  $X$  has a regular base at non-isolated points, then  $X$  is a  $T_2$ -paracompact space with a development at non-isolated points.

Sufficiency. Let  $X$  be a  $T_2$ -paracompact space with a development  $\{\mathcal{W}_i\}_{i \in \mathbb{N}}$  at non-isolated points. Since  $X$  is a  $T_2$ -paracompact space, there exists a sequence of open covers  $\{\mathcal{B}_i\}_{i \in \mathbb{N}}$  for  $X$  such that  $\mathcal{B}_{i+1}$  is a star refinement of  $\mathcal{B}_i \wedge \mathcal{W}_{i+1}$  for every  $i \in \mathbb{N}$ . We next prove that  $\{\mathcal{B}_i\}_{i \in \mathbb{N}}$  is a strong development at non-isolated points for  $X$ . For every  $x \in X - I$  and every open neighborhood  $U$  of  $x$ , there exists an  $i \in \mathbb{N}$  such that  $\text{st}(x, \mathcal{W}_i) \subset U$ . Choose a  $V \in \mathcal{B}_{i+1}$  such that  $x \in V$ . Then

$$\text{st}(V, \mathcal{B}_{i+1}) \subset \text{st}(x, \mathcal{B}_i) \subset \text{st}(x, \mathcal{W}_i) \subset U.$$

By Theorem 2.7,  $X$  has a regular base at non-isolated points.  $\square$

**Remark** We cannot omit the condition “ $T_2$ ” in Theorem 2.8. In fact, let  $X$  be the finite complement topology on  $\mathbb{N}$ . Then  $X$  is a  $T_1$ -compact and developable space, but it is not a  $T_2$ -space.

The following corollary is a complement for Lemma 2.5.

**Corollary 2.9.** *A space  $X$  has a regular base at non-isolated points if and only if  $X$  is a regular space with a development at non-isolated points  $\{\mathcal{B}_i \cup \mathcal{I}(X)\}_{i \in \mathbb{N}}$ , where  $\mathcal{B}_i$  is locally finite at non-isolated points for every  $i \in \mathbb{N}$ .*

*Proof.* Necessity. It is easy to see by the proof of necessity in Theorems 2.7 and 2.8.

Sufficiency. Let  $X$  be a regular space with a development at non-isolated points  $\{\mathcal{B}_i \cup \mathcal{I}(X)\}_{i \in \mathbb{N}}$ , where  $\mathcal{B}_i$  is locally finite at non-isolated points for every  $i \in \mathbb{N}$ . For each  $i \in \mathbb{N}$ , let

$$U_i = \{x \in X : \mathcal{B}_i \text{ is locally finite at point } x\}.$$

Then  $U_i$  is an open subset and  $\mathcal{B}_i$  is locally finite at each point of  $U_i$ . Since  $X - I \subset U_i$ ,  $X - U_i \subset I$  and  $X - U_i$  is an open subset for  $X$ . Hence  $U_i$  is an open and closed subset for  $X$ . Thus  $\mathcal{B}_i|_{U_i} = \{B \cap U_i : B \in \mathcal{B}_i\}$  is an open and locally finite family.

By Theorem 2.8, we only need to prove that  $X$  is a paracompact space. In fact, for every open cover  $\mathcal{U}$  of  $X$  and each  $i \in \mathbb{N}$ , let

$$\mathcal{V}_i = \{B \cap U_i : B \in \mathcal{B}_i \text{ and there exists an } U \in \mathcal{U} \text{ such that } B \subset U\}$$

and

$$V_i = \bigcup \mathcal{V}_i.$$

Put

$$\mathcal{V} = \left( \bigcup_{i \in \mathbb{N}} \mathcal{V}_i \right) \cup \{ \{x\} : x \in F \}, \text{ where } F = \bigcap_{i \in \mathbb{N}} (X - V_i).$$

Then  $\mathcal{V}$  is a cover for  $X$  and  $F \subset I$ . In fact, if  $x \in X - I$ , then there exists an  $U \in \mathcal{U}$  such that  $x \in U$ . Hence there exists an  $n \in \mathbb{N}$  such that  $\text{st}(x, \mathcal{B}_n) \subset U$ . Fix a  $B \in \mathcal{B}_n$  such that  $x \in B$ . Then  $B \subset U$  and  $x \in B \cap U_n \in \mathcal{V}_n$ . So  $x \in V_n$ . Then  $F$  is a closed and discrete subset for  $X$ . Hence  $\mathcal{V}$  is an open  $\sigma$ -locally finite cover and refines  $\mathcal{U}$ . By the regularity,  $X$  is a paracompact space.  $\square$

*Example 2.10.* There exists a non-regular  $T_2$ -space with a development at non-isolated points.

Let  $\mathbb{Q}, \mathbb{P}$  denote the rational numbers and the irrational numbers, respectively. Let  $X = \mathbb{R}$  and endow  $X$  with the following topology [4]: every point of  $\mathbb{P}$  is an isolated point; every point  $x \in \mathbb{Q}$  has neighborhoods of the following form:

$$B(x, n) = \{x\} \cup \{y \in \mathbb{P} : |y - x| < 1/n\}, n \in \mathbb{N}.$$

Then  $X$  is a non-regular  $T_2$ -space and the isolated points set of  $X$  is  $\mathbb{P}$ . We denote  $\mathbb{Q} = \{q_m : m \in \mathbb{N}\}$ . For any  $n, m \in \mathbb{N}$ , let

$$\mathcal{B}_{n,m} = \{B(q_m, n), \mathbb{R} - \{q_m\}\},$$

Then  $\mathcal{B}_{n,m}$  is a finite open cover for  $X$ , and  $\text{st}(q_m, \mathcal{B}_{n,m} \cup \mathcal{I}(X)) = B(q_m, n)$ . Hence  $\{\mathcal{B}_{n,m} \cup \mathcal{I}(X)\}_{n,m \in \mathbb{N}}$  is a development at non-isolated points for  $X$  and  $\mathcal{B}_{n,m}$  is locally finite for any  $n, m \in \mathbb{N}$ .

### 3. METRIZATION THEOREMS

In this section we shall discuss the metrization problems on spaces with the properties of bases at non-isolated points.

$X$  is called a *perfect space* if every open subset of  $X$  is an  $F_\sigma$ -set in  $X$ .

**Theorem 3.1.** *Let  $X$  be a space. Then the following are equivalent:*

- (1)  $X$  is metrizable;
- (2)  $X$  is a perfect space with a regular base at non-isolated points;
- (3)  $X$  is a perfect space with a strong development at non-isolated points.

*Proof.* By Theorems 1.1 and 2.7, we only need to prove (3)  $\Rightarrow$  (1).

Let  $X$  be a perfect space with a strong development at the non-isolated points  $\{\mathcal{W}_i\}_{i \in \mathbb{N}}$  of  $X$ . Then there exists a sequence of open sets  $\{G_n\}_{n \in \mathbb{N}}$  such that  $X - I = \bigcap_{n=1}^{\infty} G_n$ . For every  $n \in \mathbb{N}$ , let  $\mathcal{U}_n = \{G_n\} \cup \{\{x\} : x \in I - G_n\}$ . Then  $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$  is a sequence of open covers for  $X$ . Put  $\mathcal{V}_{2n-1} = \mathcal{W}_n$  and  $\mathcal{V}_{2n} = \mathcal{U}_n$ , for each  $n \in \mathbb{N}$ . Then  $\{\mathcal{V}_n\}_{n \in \mathbb{N}}$  is a strong development for  $X$ , and  $X$  is metrizable by [6, Theorem 5.4.2].  $\square$

**Remark** By Example 1.2, we see the condition “ $X$  is perfect” in (2) and (3) of Theorem 3.1 cannot be omitted, although clearly it can be replaced with the condition that  $I(X)$  is an  $F_\sigma$ -set.

**Definition 3.2.** Let  $\mathcal{B} = \bigcup_{i \in \mathbb{N}} \mathcal{B}_i$  be a base for space  $X$ .  $\mathcal{B}$  is called  *$\sigma$ -locally finite at non-isolated points*, if for every  $i \in \mathbb{N}$ ,  $\mathcal{B}_i$  is locally finite at non-isolated points for  $X$ .

Similarly, we can define the notion of spaces with a  $\sigma$ -discrete base at non-isolated points.

**Definition 3.3.** Let  $\mathcal{B}$  be a family of subsets of  $X$ . For every  $x \in X$ ,  $\mathcal{B}$  is called *hereditarily closure-preserving at  $x$*  if, for any  $H(B) \subset B \in \mathcal{B}$ ,  $x \in \overline{\cup\{H(B) : B \in \mathcal{B}\}}$ , then  $x \in \cup\{\overline{H(B)} : B \in \mathcal{B}\}$ .  $\mathcal{B}$  is called *a hereditarily closure-preserving collection* for  $X$  if, for every  $x \in X$ ,  $\mathcal{B}$  is hereditarily closure-preserving at  $x$ .

It is easy to verify that a collection is hereditarily closure preserving if and only if it is hereditarily closure preserving at non-isolated points.

**Lemma 3.4.** *Let  $\mathcal{B}$  be locally finite at non-isolated points for  $X$ . Then  $\mathcal{B}$  is hereditarily closure-preserving.*

*Proof.* Let  $\mathcal{B} = \{B_\alpha : \alpha \in \Gamma\}$ . For every  $\alpha \in \Gamma$ , choose  $H_\alpha \subset B_\alpha$ . We can assume  $x \in X - I$  and denote  $\mathcal{H} = \{H_\alpha\}_{\alpha \in \Gamma}$ . If  $x \in \overline{\cup\mathcal{H}}$ , then there exists an open neighborhood  $U(x)$  of  $x$  such that the set of all members of  $\{H_\alpha\}_{\alpha \in \Gamma}$  that meet  $U(x)$  is finite because  $\{H_\alpha\}_{\alpha \in \Gamma}$  is locally finite at non-isolated points. we denote these finite elements by  $H_{\alpha_1}, H_{\alpha_2}, \dots, H_{\alpha_n}$ . Since

$$\overline{\cup\mathcal{H}} = \overline{\cup(\mathcal{H} - \{H_{\alpha_i} : i \leq n\})} \cup \overline{\cup\{H_{\alpha_i} : i \leq n\}}, \text{ and}$$

$$U(x) \cap (\cup(\mathcal{H} - \{H_{\alpha_i} : i \leq n\})) = \emptyset,$$

we have  $x \in \overline{\cup\{H_{\alpha_i} : i \leq n\}}$ . Hence  $x \in \overline{\cup\mathcal{H}}$ . □

**Lemma 3.5.** [5] *A regular space  $X$  is metrizable if and only if  $X$  has a  $\sigma$ -hereditarily closure-preserving base.*

**Lemma 3.6.** *Let  $X$  be a regular space. Then the following conditions are equivalent:*

- (1)  $X$  is metrizable;
- (2)  $X$  has a base which is  $\sigma$ -discrete at non-isolated points;
- (3)  $X$  has a base which is  $\sigma$ -locally finite at non-isolated points.

*Proof.* It is easy to see by Theorem 1.1, Lemmas 3.4 and 3.5 □

Let  $X$  be a topological space and  $\tau(X)$  its topology.  $g : \mathbb{N} \times X \rightarrow \tau(X)$  is called a  $g$ -function if, for any  $x \in X$  and  $n \in \mathbb{N}$ ,  $x \in g(n, x)$ . A space  $X$  is called a  $\beta$ -space [11] if there exists a  $g$ -function such that, for every  $x \in X$  and sequence  $\{x_n\}$  in  $X$ , if  $x \in g(n, x_n)$  for each  $n \in \mathbb{N}$ , then  $\{x_n\}$  has a cluster point in  $X$ . Obviously every developable space is a  $\beta$ -space.

**Theorem 3.7.** *A space  $X$  is metrizable if and only if  $X$  is a  $\beta$ -space with a regular base at non-isolated points.*

*Proof.* We only need to prove the sufficiency. Let  $X$  be a  $\beta$ -space with a regular base at non-isolated points. By Theorem 3.1, it suffices to prove that  $I(X)$  is an  $F_\sigma$ -set. Suppose  $g$  is a  $g$ -function satisfying the above definition of  $\beta$ -spaces. Since  $X$  has a regular base at non-isolated points,  $X$  has a regular base at non-isolated points  $\mathcal{B} = (\bigcup_{n \in \mathbb{N}} \mathcal{B}_n) \cup \mathcal{I}(X)$  satisfying Lemma 2.5, where  $\mathcal{B}_n$  is locally finite at non-isolated points and  $\mathcal{B}_{n+1} \cup \mathcal{I}(X)$  refines  $\mathcal{B}_n \cup \mathcal{I}(X)$  for each  $n \in \mathbb{N}$ . For each  $n \in \mathbb{N}$  and  $x \in X - I$ , put

$$b(n, x) = \cap\{B \in \mathcal{B}_n : x \in B\}.$$

Then  $\{b(n, x)\}_{n \in \mathbb{N}}$  is a local base for  $x \in X - I$ . For each  $n \in \mathbb{N}$ , put

$$h(n, x) = (\cap\{g(i, x) : i \leq n\}) \cap b(n, x), \quad x \in X - I;$$

$$H_n = \cup\{h(n, x) : x \in X - I\}.$$

Then  $X - I \subset H_n$  and  $H_n$  is an open subset for  $X$ . We next prove  $X - I = \bigcap_{n \in \mathbb{N}} H_n$ . Let  $x \in \bigcap_{n \in \mathbb{N}} H_n$ . Then there exists some point  $x_n \in X - I$  such that  $x \in h(n, x_n)$  for each  $n \in \mathbb{N}$ . Since  $X$  is a  $\beta$ -space and  $x \in g(n, x_n)$ ,  $\{x_n\}$  has a cluster point in  $X$ . Let  $y$  be a cluster point of  $\{x_n\}$ . Then  $y \in X - I$  and  $b(n, y)$  is an open neighborhood of  $y$ . Without loss of generality, we can assume  $x_{n_i} \in b(i, y)$  for each  $i \in \mathbb{N}$ . We will show that  $b(i, x_{n_i}) \subset b(i, y)$ . If not, choose a point  $z \in b(i, x_{n_i}) - b(i, y)$ , then there exists a  $B \in \mathcal{B}_i$  such that  $y \in B$  and  $z \notin B$ . Since  $x_{n_i} \in b(i, y) \subset B$ ,  $z \in b(i, x_{n_i}) \subset B$ , a contradiction. Hence

$$x \in \bigcap_{i \in \mathbb{N}} h(n_i, x_{n_i}) \subset \bigcap_{i \in \mathbb{N}} h(i, x_{n_i}) \subset \bigcap_{i \in \mathbb{N}} b(i, y) = \{y\},$$

i.e.,  $x = y \in X - I$ . Thus  $X - I = \bigcap_{n \in \mathbb{N}} H_n$ , and  $I$  is an  $F_\sigma$ -set for  $X$ . By Theorem 3.1,  $X$  is metrizable.  $\square$

**Remark** The Stone-Ćech compactification  $\beta\mathbb{N}$  of  $\mathbb{N}$  is a  $\beta$ -space, but it is not a perfect space [6, Corollary 3.6.15]; Sorgenfrey line is a perfect space, but it is not a  $\beta$ -space [11, Example 4.4]. Hence, Theorem 3.1 and Theorem 3.7 are independent each other.

#### 4. RELATIONS WITH GENERALIZED METRIZABLE SPACES

**Definition 4.1.** [14] Let  $X$  be a topological space and let  $A$  be a subset of  $X$ . The *discretization* of  $X$  by  $A$  is the space whose topology is generated by the base  $\{U : U \text{ is an open subset of } X\} \cup \{\{x\} : x \in A\}$ . It is denoted by  $X_A$  in [6, Example 5.1.22]. We say that a space  $Y$  is a *discretization* of  $X$  if  $Y = X_A$  for some  $A \subset X$ .

**Theorem 4.2.** *Let  $X$  be a metric space. If  $A \subset X$  and  $X_A$  is the discretization of  $X$  by  $A$ , then  $X_A$  has a regular base at non-isolated points.*

*Proof.* Since  $X$  is a metric space,  $X$  has a regular base  $\mathcal{B}_1$ . Let  $\mathcal{B} = \mathcal{B}_1 \cup \{\{x\} : x \in A\}$ . Obviously,  $\mathcal{B}$  is a regular base at non-isolated points for  $X_A$ .  $\square$

**Remark** If a space  $X$  with a regular base at non-isolated points, then is it a discretizable space of a metric space? The answer is negative, see Example 4.3. Recall that  $X$  is said to have a  $G_\delta$ -diagonal if there exists a sequence  $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$  of open covers such that  $\{x\} = \bigcap_{n \in \mathbb{N}} \text{st}(x, \mathcal{U}_n)$  for every  $x \in X$ .

*Example 4.3.* There exists a space  $Y$  having a regular base at non-isolated points. However,  $Y$  is not a discretization of a metric space.

Let  $X$  be the Michael line in Example 1.2, and denote it by  $X_B$ . Let  $X^*$  be a copy of  $X_B$  and  $f : X_B \rightarrow X^*$  a homeomorphic map. Put  $Z = X_B \oplus X^*$  and let  $g : Z \rightarrow Y$  be a quotient map by identifying  $\{x, f(x)\}$  to a point for each  $x \in X_B \setminus B$  in  $Z$ . Then  $Y$  is a quotient space.

By [16], it is easy to see  $Y$  has no  $G_\delta$ -diagonal. Since the discretization of a metric space has a  $G_\delta$ -diagonal,  $Y$  is not a discretization of a metric space. We next prove that  $Y$  has a regular base at non-isolated points.



Put  $\mathcal{I} = \{\{x\} : x \in B\}$  and let  $\mathcal{B}$  be a regular base of  $\mathbb{I}$  with the Euclidean topology. Then  $\mathcal{B} \cup \mathcal{I}$  is a regular base at non-isolated points for  $X_B$ . Hence  $f(\mathcal{B}) \cup f(\mathcal{I})$  is a regular base at non-isolated points for  $X^*$ . Then  $\mathcal{G} = \{g(B \cup f(B)) : B \in \mathcal{B}\} \cup \mathcal{I} \cup f(\mathcal{I})$  is a regular base at non-isolated points for  $Y$ .

Indeed, it is easy to see that  $\mathcal{G}$  is a base for  $Y$ . For every  $y \in Y - I(Y)$  and each open neighborhood  $U$  of  $y$  in  $Y$ , there exists a point  $x \in X_B$  such that  $g(x) = y$ . Then  $g(f(x)) = y$ , and  $x, f(x) \in g^{-1}(U)$ . Since

$$\mathcal{B}_0 = \mathcal{B} \cup f(\mathcal{B}) \cup \mathcal{I} \cup f(\mathcal{I})$$

is a regular base at non-isolated points for  $Z$ , there exist open neighborhoods  $V_x, V_{f(x)} \subset g^{-1}(U)$  of  $x, f(x)$  in  $Z$  respectively such that the set of all members of  $\mathcal{B}_0$  that meet  $V_x$  and  $Z - g^{-1}(U)$  is finite, and the set of all members of  $\mathcal{B}_0$  that meet  $V_{f(x)}$  and  $Z - g^{-1}(U)$  is also finite. Since  $f$  is a homeomorphic map, there exists a  $B \in \mathcal{B}$  such that  $x \in B \subset V_x$  and  $f(x) \in f(B) \subset V_{f(x)}$ . Then  $g(x) = y \in g(B \cup f(B)) \subset U$ . Since the set of all members of  $\mathcal{B}_0$  that meet  $B \cup f(B)$  and  $Z - g^{-1}(U)$  is finite. If  $V \in \mathcal{B}_0$ , then  $g^{-1}(g(V)) = V$ , hence the set of all members of  $\mathcal{G}$  that meet  $g(B \cup f(B))$  and  $Y - U$  is finite. Thus  $Y$  has a regular base at non-isolated points.

**Definition 4.4.** [14] An *ortho-base*  $\mathcal{B}$  for  $X$  is a base of  $X$  such that either  $\cap \mathcal{A}$  is open in  $X$  or  $\cap \mathcal{A} = \{x\} \notin \mathcal{I}(X)$  and  $\mathcal{A}$  is a neighborhood base at  $x$  in  $X$  for each  $\mathcal{A} \subset \mathcal{B}$ . A space  $X$  is a *proto-metrizable space* if it is a paracompact space with an ortho-base.

Recall that a space  $X$  is called a  $\gamma$ -space if there exists a  $g$ -function  $g(n, x)$  for  $X$  satisfying for each  $x \in X$  and sequences  $\{x_n\}, \{y_n\}$  if  $x_n \in g(n, y_n)$  and  $y_n \in g(n, x)$  for each  $n \in \mathbb{N}$ , then  $x_n \rightarrow x$ .

**Theorem 4.5.** *If a space  $X$  has a regular base at non-isolated points, then  $X$  is:*

- (1) *a proto-metrizable space, and*
- (2) *a  $\gamma$ -space.*

*Proof.* (1) By Lemma 2.6 and Theorem 2.7,  $X$  is a paracompact space. Also,  $X$  has an ortho-base by [7, Theorem 3.4]. Hence  $X$  is a proto-metrizable space.

(2) To prove part (2), for each  $n \in \mathbb{N}$  and  $x \in X$  define a function  $g : \mathbb{N} \times X \rightarrow \tau(X)$  as follows: if  $x \in I$ , then  $g(n, x) = \{x\}$ ; if  $x \in X - I$ , then  $g(n, x) = b(n, x)$ , where  $b(n, x)$  is the same as in the proof in Theorem 3.7. Then  $\{g(n, x)\}_{n \in \mathbb{N}}$  is a decreasing and open neighborhood base of  $x$ , and if  $y \in g(n, x)$ , then  $g(n, y) \subset g(n, x)$ . For each  $x \in X$  and sequences  $\{x_n\}, \{y_n\}$ , if  $x_n \in g(n, y_n)$  and  $y_n \in g(n, x)$  for each  $n \in \mathbb{N}$ , then  $x_n \in g(n, y_n) \subset g(n, x)$ , thus  $x_n \rightarrow x$ . Hence  $X$  is a  $\gamma$ -space.  $\square$

*Example 4.6.* There exists a proto-metrizable space which has no regular base at non-isolated points.

The proto-metrizable but non- $\gamma$ -space described in Section 3 in [9] works.

**Remark** From the discussion above, it can be seen that spaces with a regular base at non-isolated points are strictly between the discretizations of metrizable spaces and proto-metrizable spaces.

**Corollary 4.7.** *Let  $X$  have a  $G_\delta$ -diagonal. Then the following conditions are equivalent:*

- (1)  *$X$  is a discretizations of a metrizable space;*
- (2)  *$X$  has a regular base at non-isolated points;*

(3)  $X$  is a proto-metrizable space.

*Proof.* By Theorems 4.2 and 4.5, we have  $(1) \Rightarrow (2) \Rightarrow (3)$ . By [9, Theorem 3.1], it can be obtained  $(3) \Rightarrow (1)$ .  $\square$

The condition “ $G_\delta$ -diagonal” cannot be omitted in Corollary 4.7 by Example 4.3.

**Question 4.8.** *Under what conditions a proto-metrizable space has a regular base at non-isolated points?*

**Remark** Since a proto-metrizable space is a paracompact space, Theorem 2.8 is an answer for Question 4.8. However, we expect a simpler answer.

**Definition 4.9.** Let  $\mathcal{B}$  be a base of a space  $X$ .  $\mathcal{B}$  is *point-regular* [1] (*point-regular at non-isolated points* [7], resp.) for  $X$ , if for each (non-isolated, resp.) point  $x \in X$  and  $x \in U$  with  $U$  open in  $X$ ,  $\{B \in \mathcal{B} : x \in B \not\subset U\}$  is finite.

Obviously, every regular base at non-isolated points is a point-regular base at non-isolated points. In [7], it is proved that a space  $X$  has a point-regular base at non-isolated points if and only if  $X$  is an open, boundary-compact image of a metric space. On the other hand, a space  $X$  is an open, boundary-compact,  $s$ -image of a metric space if and only if  $X$  has a point-countable base which is point-regular at non-isolated points. The following question is posed in [7, Question 5.1]:

**Question 4.10.** [7, Question 5.1] *Let a space  $X$  have a point-countable base. If  $X$  has a point-regular base at non-isolated points, is  $X$  an open, boundary-compact,  $s$ -image of a metric space?*

Next, we give an affirmative answer for Question 4.10.

A space  $X$  is called *metalindelöf* if every open cover of  $X$  has a point-countable open refinement.

**Theorem 4.11.** *The following are equivalent for a space  $X$ :*

- (1)  $X$  has a point-countable base, and has a point-regular base at non-isolated points;
- (2)  $X$  has a point-countable base which is point-regular at non-isolated points;
- (3)  $X$  is an open boundary-compact,  $s$ -image of a metric space;
- (4)  $X$  is an open  $s$ -image of a metric space, and is an open boundary-compact image of a metric space;
- (5)  $X$  is a metalindelöf space with a point-regular base at non-isolated points.

*Proof.* It is proved in [7] that if  $\mathcal{P}$  is a point-regular base at non-isolated points for a space  $X$ , then we can assume that  $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$  satisfies the following conditions:

- (a)  $\mathcal{P}_n$  is an open cover and is point-finite at non-isolated points;
- (b)  $\{\mathcal{P}_n\}$  is a development at non-isolated points for  $X$ .

$(1) \Rightarrow (2)$ . Suppose that  $X$  has a point-countable base  $\mathcal{B}$ , and suppose that  $X$  has a point-regular base at non-isolated points  $\mathcal{P}$ . We can assume that  $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$  satisfies the conditions (a) and (b). For each  $n \in \mathbb{N}$ , put

$$\begin{aligned} \mathcal{B}' &= \{B \in \mathcal{B} : B \not\subset I(X)\}; \\ \mathcal{V}_n(B) &= \{P \in \mathcal{P}_n : B \subset P\}, \quad \forall B \in \mathcal{B}'; \end{aligned}$$

$$\begin{aligned}\hat{P} &= \cup\{B \in \mathcal{B}' : P \in \mathcal{V}_n(B)\}, \quad \forall P \in \mathcal{P}_n; \\ \hat{\mathcal{P}}_n &= \{\hat{P} : P \in \mathcal{P}_n\}.\end{aligned}$$

Then  $\hat{\mathcal{P}}_n$  is point-countable. In fact, if  $x \in \hat{P} \in \hat{\mathcal{P}}_n$ , then there is  $B' \in \mathcal{B}'$  such that  $x \in B'$  and  $P \in \mathcal{V}_n(B')$ . Since  $\{B \in \mathcal{B}' : x \in B\}$  is countable, and each  $\mathcal{V}_n(B)$  is finite for each  $B \in \mathcal{B}'$  by the condition (a), it follows that  $\{P \in \mathcal{V}_n(B) : x \in B \in \mathcal{B}'\}$  is countable.

Put

$$\hat{\mathcal{P}} = (\bigcup_{n \in \mathbb{N}} \hat{\mathcal{P}}_n) \cup \mathcal{I}(X).$$

Then  $\hat{\mathcal{P}}$  is point-countable. If  $x \in U - I$  with  $U$  open in  $X$ , then there is  $m \in \mathbb{N}$  such that  $x \in \text{st}(x, \mathcal{P}_m) \subset U$  by the condition (b). Take  $P \in \mathcal{P}_m$  with  $x \in P$ , then there is  $B \in \mathcal{B}'$  such that  $x \in B \subset P$ , thus  $P \in \mathcal{V}_m(B)$ , and  $x \in B \subset \hat{P} \subset P \subset U$ . So  $\hat{\mathcal{P}}$  is a base for  $X$ . Finally, it is easy to see that  $\hat{\mathcal{P}}$  is point-regular at non-isolated points by  $\hat{P} \subset P$  for each  $P \in \mathcal{P}$ .

(2)  $\Rightarrow$  (3) by [7, Corollary, 3.2]. (3)  $\Rightarrow$  (4) is obvious. And (4)  $\Rightarrow$  (5) by [7, Theorem, 3.1].

(5)  $\Rightarrow$  (1). Let  $X$  be a metalindelöf space with a point-regular base at non-isolated points. As in the proof of (1)  $\Rightarrow$  (2), there is a sequence  $\{\mathcal{P}_n\}$  of open covers of  $X$  such that  $\{\mathcal{P}_n\}$  is a development at non-isolated points for  $X$ . For each  $n \in \mathbb{N}$ , let  $\mathcal{B}_n$  be a point-countable open refinement of  $\mathcal{P}_n$ . And put

$$\mathcal{B} = (\bigcup_{n \in \mathbb{N}} \mathcal{B}_n) \cup \mathcal{I}(X).$$

Then  $\mathcal{B}$  is a point-countable base for  $X$ . In fact, if a non-isolated point  $x \in U$  with  $U$  open in  $X$ , then there is  $n \in \mathbb{N}$  such that  $\text{st}(x, \mathcal{P}_n) \subset U$ . Take  $B \in \mathcal{B}_n$  with  $x \in B$ , then  $x \in B \subset \text{st}(x, \mathcal{B}_n) \subset \text{st}(x, \mathcal{P}_n) \subset U$ .  $\square$

By Theorem 4.11, the following is obtained.

**Corollary 4.12.** *Every space with a regular base at non-isolated points has a point-countable base.*

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